Distance Computation Between Extended Quadratic Complexes

Christian Lennerz

June 17, 2002

Conics, Quadrics and Quadratic Complexes

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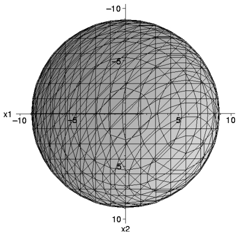
  for a vector $\mathbf{a} \in \mathbb{R}^3$ and symmetric matrix $\mathbf{A} \in \mathbb{R}^{3\times 3}$.

- A conic is explicitly given as the following point set:

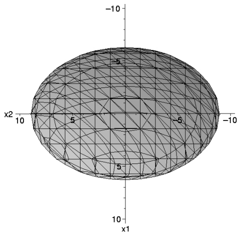
$$\{\mathbf{p} \in \mathbb{R}^3 \,|\, \mathbf{p} \;=\; \mathbf{c} + r(t)\mathbf{u} + s(t)\mathbf{v}\},$$

  where $(r, s) \in \{(\cos, \sin), (\cosh, \sinh), (\mathrm{id}, \mathrm{id}^2), (\mathrm{id}, 0)\}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ with $\mathbf{u}^T\mathbf{v} = 0$.

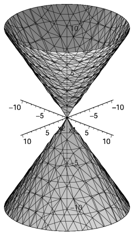
Examples of Quadrics



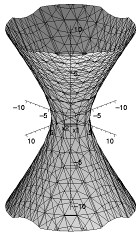
Sphere



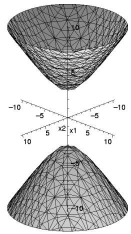
Ellipsoid



Cone



One-Sheet-
Hyperboloid



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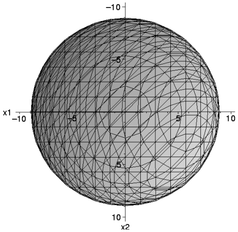
Examples of Quadrics



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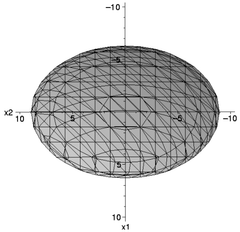

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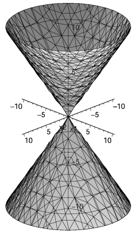

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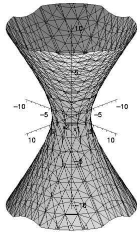

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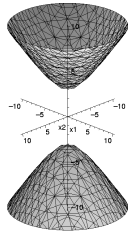
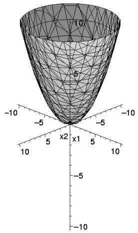
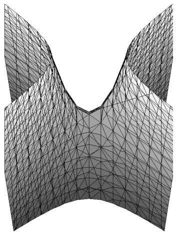


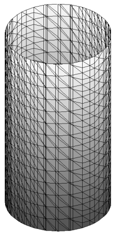
Two-Sheet
Hyperboloid



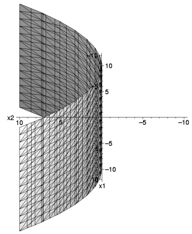
Elliptic
Paraboloid



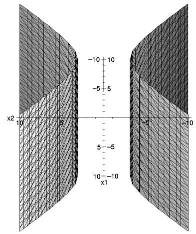
Hyperbolic
Paraboloid



Cylinder

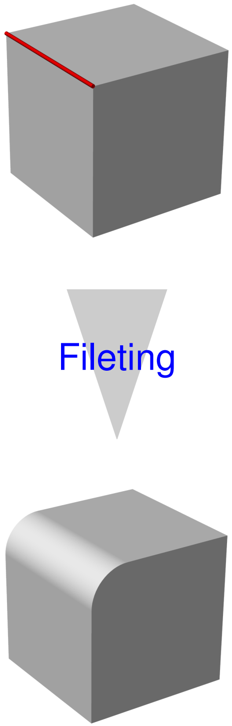


Parabolic
Cylinder



Hyperbolic
Cylinder

Examples of Quadratic Complexes in CAD



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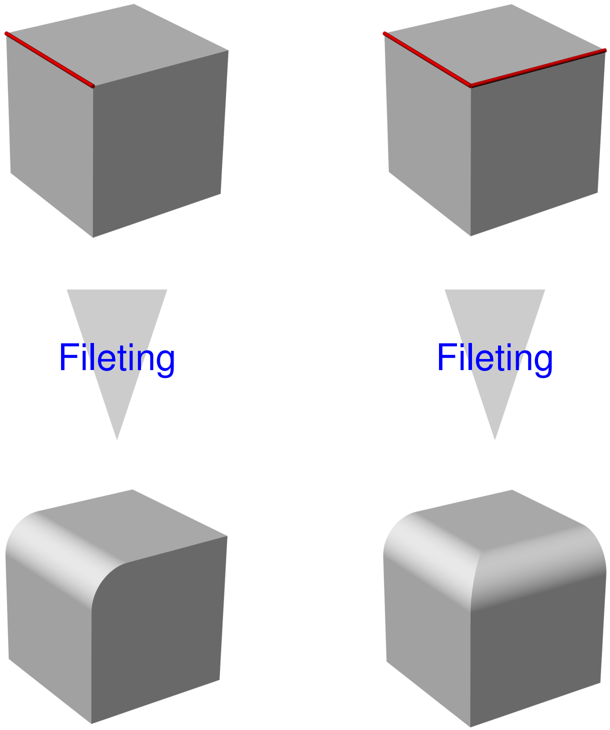

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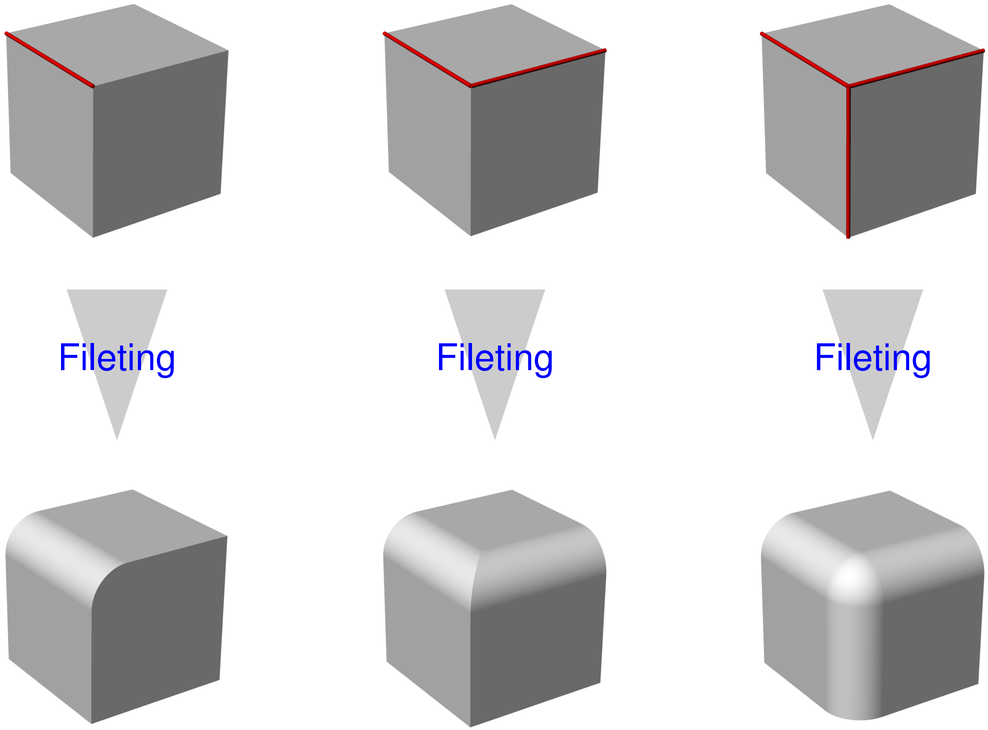
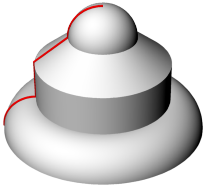
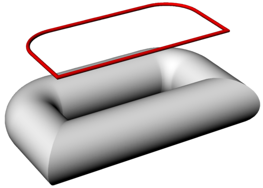


Some Limitations

- Typical CAD-operations on circular profile curves lead to torus patches:



Revolving



Tubing

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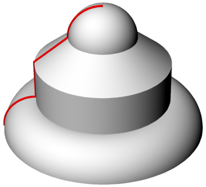

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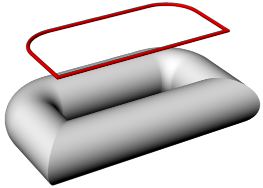
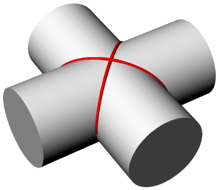
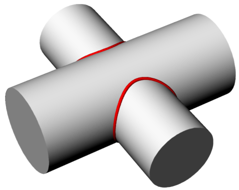


Tubing

- The class of quadratic complexes is not closed under BOOLEAN-operations:



Union (same radii)



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Extended Quadratic Complexes

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Due to these limitations one has to think about extending the class of quadratic complexes:

- Geometrical embedding of faces on torus surfaces,
- Approximation of non-conic intersection curves by
  - conic arcs or
  - nurbs curves,
- Representation of non-conic intersection curves by their exact parameterization.

Classification of Quadrics by Normal Forms

From the implicit representation of a general quadric, we can derive the following normal forms by applying coordinate transformations:

Central Surfaces: $\det(\mathbf{A}) \neq 0$	
Ellipsoid / Hyperboloids	$\mathbf{a} = \mathbf{0} \quad a_0 \neq 0$
Cone	$\mathbf{a} = \mathbf{0} \quad a_0 = 0$

Non-Central Surfaces: $\det(\mathbf{A}) = 0$		
Paraboloids	$A_3 = 0$	$a_3 \neq 0 \quad a_0 = 0$
Elliptic /Hyperbolic Cylinder	$A_3 = 0$	$\mathbf{a} = \mathbf{0} \quad a_0 \neq 0$
Parabolic Cylinder	$A_1 = A_3 = 0$	$a_1 \neq 0 \quad a_0 = 0$

Examples of Quadrics in Normal Form

Central Surfaces:

Example 1 (Ellipsoid) *An ellipsoid in normal form is given as:*

$$\mathbf{x}^T\mathbf{A}\mathbf{x} + a_0 \; = \; 0, \quad \textit{with} \;\; \mathbf{A} = \begin{bmatrix} 1/r_1^2 & 0 & 0 \\ 0 & 1/r_2^2 & 0 \\ 0 & 0 & 1/r_3^2 \end{bmatrix}, \; a_0 = -1.$$

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Non-Central Surfaces:

Example 2 (Elliptic Cylinder) *An elliptic cylinder in normal form is given as:*

$$\mathbf{x}^T \mathbf{A} \mathbf{x} + a_0 \ = \ 0, \quad \textit{with} \ \mathbf{A} = \begin{bmatrix} 1/r_1^2 & 0 & 0 \\ 0 & 1/r_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ a_0 = -1.$$

The Distance Computation Problem

Definition 1 (Distance Computation Problem)
Given two quadratic complexes $\mathbf{C}_1$*,* $\mathbf{C}_2$*. The distance computation problem is to determine the global minimum of the distance function* δ *between the respective point sets, together with a pair of witness points i.e.*

(i) the value $\delta^* := \delta(\mathbf{C}_1, \mathbf{C}_2)$*,*

(ii) a pair of points $(\mathbf{p}, \mathbf{q})$*, s.t.* $\delta^* = \delta(\mathbf{p}, \mathbf{q})$*,*

where δ *denotes the (quadratic)* EUCLIDEAN *distance function between two points or set of points, respectively.*

Closest Points Between Faces

Let f_1 and f_2 be disjoint faces of quadratic complexes that are embedded on the quadratic surfaces Q_1 and Q_2, where

$$Q_1 := \{\mathbf{x} \mid \mathbf{x}^T \mathbf{A} \mathbf{x} + 2\mathbf{a}^T \mathbf{x} + a_0 = 0\},$$
$$Q_2 := \{\mathbf{y} \mid \mathbf{y}^T \mathbf{B} \mathbf{y} + 2\mathbf{b}^T \mathbf{y} + b_0 = 0\}.$$

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If $(\mathbf{p}_1, \mathbf{p}_2)$ is a pair of closest points between f_1 and f_2, then either

(i) $(\mathbf{p}_1, \mathbf{p}_2)$ is an extremum of the distance function between Q_1 and Q_2, i.e. there are $\alpha, \beta \in \mathbb{R},\ \alpha, \beta \neq 0$ s.t.

$$\mathbf{n}(\mathbf{p}_1) = \alpha(\mathbf{p}_2 - \mathbf{p}_1) \qquad \mathbf{n}(\mathbf{p}_2) = \beta(\mathbf{p}_1 - \mathbf{p}_2),$$

where $\mathbf{n}(\mathbf{p}_i)$ denotes the normal of Q_i in $\mathbf{p}_i$, or

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If $(\mathbf{p}_1, \mathbf{p}_2)$ is a pair of closest points between f_1 and f_2, then either

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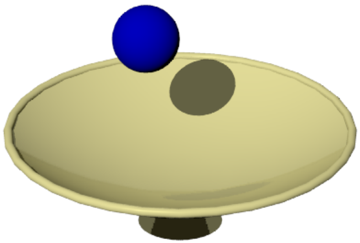
where $\mathbf{n}(\mathbf{p}_i)$ denotes the normal of Q_i in $\mathbf{p}_i$, or

(ii) $\mathbf{p}_1$ or $\mathbf{p}_2$ lies on the boundary of the face f_1 or f_2, respectively.

Distance Between Quadric Patches (Case I)



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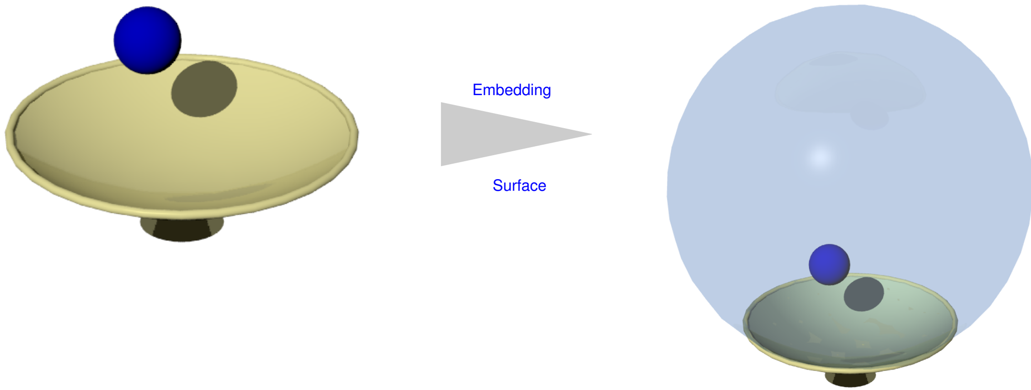

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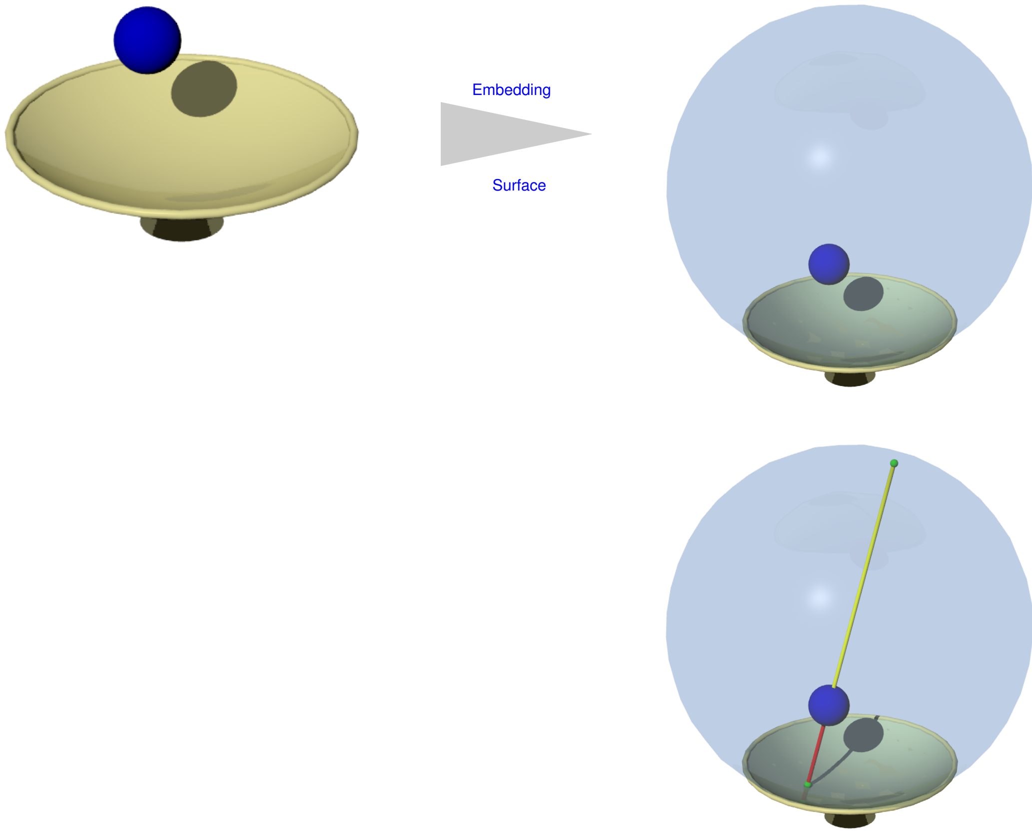

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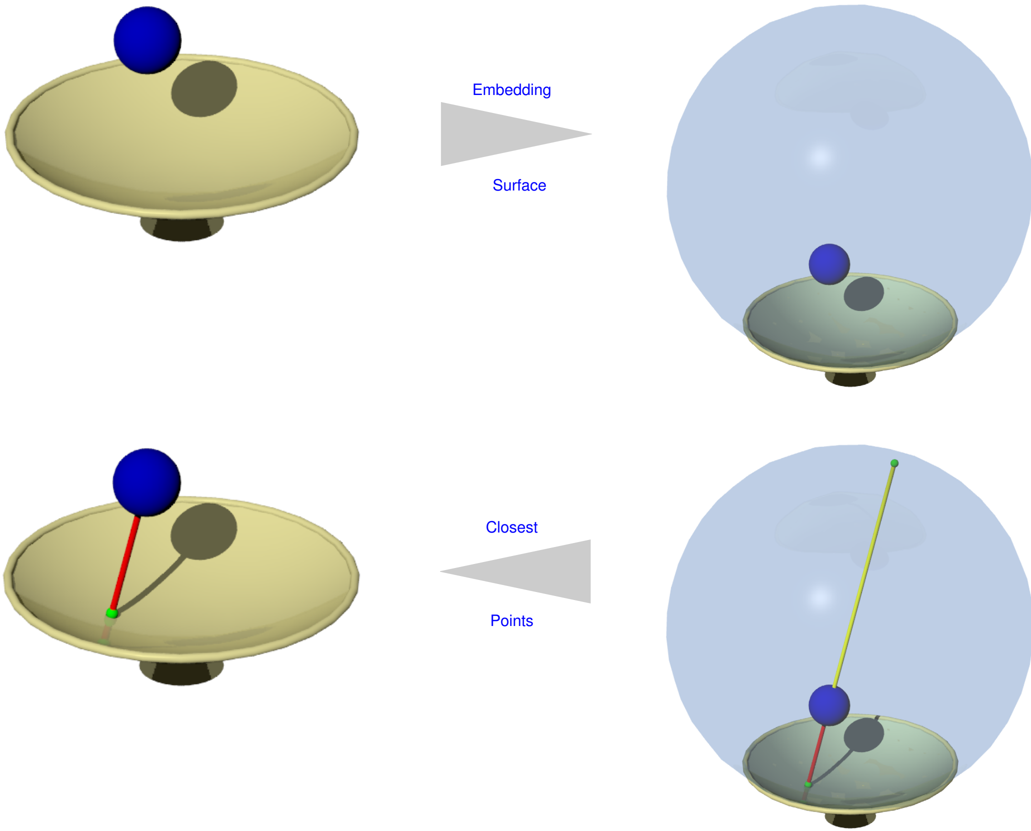
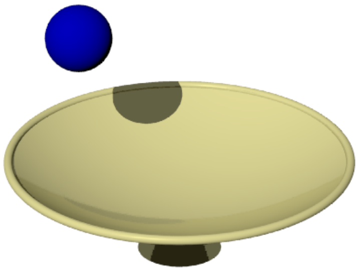


Distance Between Quadric Patches (Case II)



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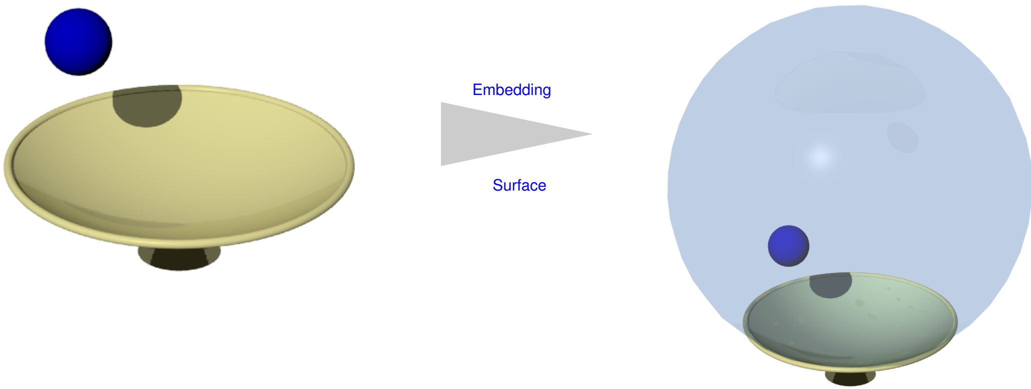

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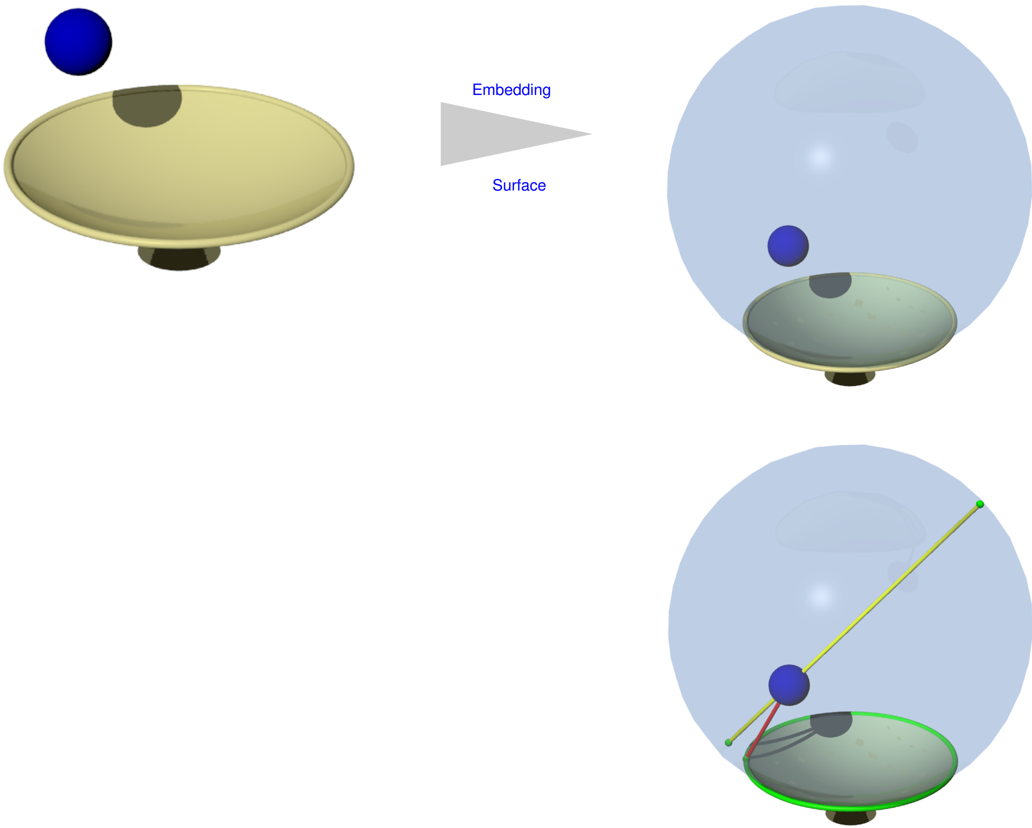

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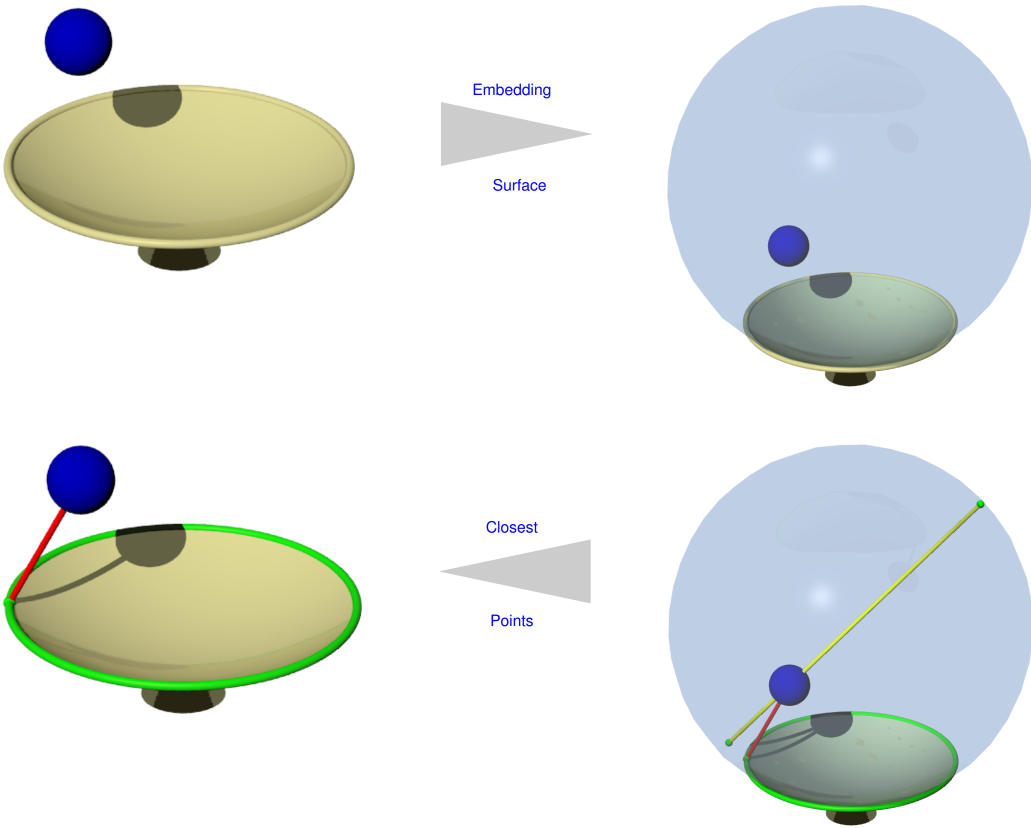

A Generic Algorithm

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$\textsc{EntityDistance}(E_1, E_2)$
(1) $\left[\mathrm{isDisjoint}, (\mathbf{p}_1, \mathbf{p}_2)\right] \leftarrow \textsc{Intersect}(E_1, E_2)$
(2) **if** $\mathrm{isDisjoint} = \mathrm{false}$
(3) **return** $\left[0, (\mathbf{p}_1, \mathbf{p}_2)\right]$

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(2) **if** $\mathtt{isDisjoint} = \mathtt{false}$
(3) **return** $\left[0, (\mathbf{p}_1, \mathbf{p}_2)\right]$
(4) $\delta_G \leftarrow \infty$
(5) **while** $\left[\delta, (\mathbf{q}_1, \mathbf{q}_2)\right] \leftarrow \textsc{Extrema}(E_1, E_2)$
(6) **if** $(\mathbf{q}_1 \in E_1)$**and** $(\mathbf{q}_2 \in E_2)$
(7) **if** $\delta < \delta_G$
(8) $\delta_G \leftarrow \delta, \quad (\mathbf{p}_1, \mathbf{p}_2) \leftarrow (\mathbf{q}_1, \mathbf{q}_2)$

A Generic Algorithm

```
EntityDistance(E1, E2)
    [isDisjoint, (p1, p2)] ← Intersect(E1, E2)
    if isDisjoint = false
        return [0, (p1, p2)]
    δG ← ∞
    while [δ, (q1, q2)] ← Extrema(E1, E2)
        if (q1 ∈ E1) and (q2 ∈ E2)
            if δ < δG
                δG ← δ,   (p1, p2) ← (q1, q2)
    foreach subentity E of E1
        [δ, (q1, q2)] ← EntityDistance(E, E2)
        if δ < δG
            δG ← δ,   (p1, p2) ← (q1, q2)
```

A Generic Algorithm

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EntityDistance(E_1, E_2)
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δ_G ← ∞
while [δ, (q_1, q_2)] ← Extrema(E_1, E_2)
    if (q_1 ∈ E_1) and (q_2 ∈ E_2)
        if δ < δ_G
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foreach subentity E of E_1
    [δ, (q_1, q_2)] ← EntityDistance(E, E_2)
    if δ < δ_G
        δ_G ← δ,    (p_1, p_2) ← (q_1, q_2)
foreach subentity E of E_2
    [δ, (q_1, q_2)] ← EntityDistance(E_1, E)
    if δ < δ_G
        δ_G ← δ,    (p_1, p_2) ← (q_1, q_2)
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foreach subentity E of E2
    [δ, (q1, q2)] ← EntityDistance(E1, E)
    if δ < δG
        δG ← δ,   (p1, p2) ← (q1, q2)
return [δG, (p1, p2)]
```

Degree Complexity of the Polynomial Systems

Theorem 1 (General Quadratic Complexes)

- *The distance between two faces of quadratic complexes can be computed by solving systems of univariate and bivariate polynomials in which the degree of every variable is at most 6.*
- *These systems can be solved by finding the roots of univariate polynomials of a degree of at most 24.*

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Theorem 2 (Natural Quadratic Complexes)
The distance between two faces embedded on natural quadrics and trimmed by natural conics can be computed by solving univariate polynomials of a degree of at most 8.

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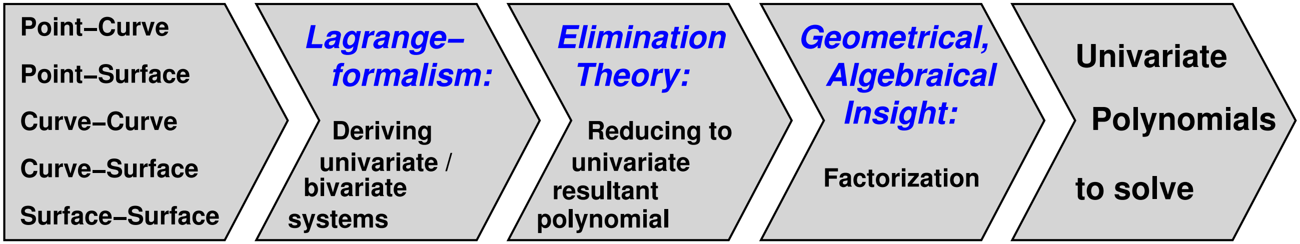
Theorem 2 (Natural Quadratic Complexes)
The distance between two faces embedded on natural quadrics and trimmed by natural conics can be computed by solving univariate polynomials of a degree of at most 8.

Remark 1 (Torus)
If one extends the classes by the torus, the results remain valid. The distance to any other surface or curve can be computed by considering its main circle.

Overview of the Approach



The Point-Surface Case

The LAGRANGE-formalism for the point-surface problem, gives:

$$\mathcal{L}(\mathbf{x};\alpha) \;=\; (\mathbf{x}-\mathbf{p})^2 + \alpha(\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} + 2\mathbf{a}^\mathsf{T}\mathbf{x} + a_0),$$

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$$\text{(i)} \quad \frac{\partial \mathcal{L}(.)}{\partial \mathbf{x}} = 0 \iff \alpha(\mathbf{A}\mathbf{x} + \mathbf{a}) = \mathbf{p} - \mathbf{x},$$

$$\text{(ii)} \quad \frac{\partial \mathcal{L}(.)}{\partial \alpha} = 0 \iff \mathbf{x}^\mathsf{T} A\mathbf{x} + 2\mathbf{a}^\mathsf{T}\mathbf{x} + a_0 = 0.$$

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$$\mathcal{L}(\mathbf{x};\alpha) \; = \; (\mathbf{x}-\mathbf{p})^2 + \alpha(\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} + 2\mathbf{a}^\mathsf{T}\mathbf{x} + a_0),$$

$$\text{(i)} \quad \frac{\partial \mathcal{L}(.)}{\partial \mathbf{x}} = 0 \iff \alpha(\mathbf{A}\mathbf{x} + \mathbf{a}) = \mathbf{p} - \mathbf{x},$$

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From the first LAGRANGE-condition, we can derive:

$$\mathbf{x} \; = \; (\mathbf{E} + \alpha\mathbf{A})^{-1}(\mathbf{p} - \alpha\mathbf{a}) \; =: \; \mathbf{D}_\alpha^{-1}\mathbf{p}_\alpha.$$

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Substituting $\mathbf{x}$ in the second equation gives the univariate system:

$$f(\alpha) \;=\; \mathbf{p}_\alpha^\top \overline{\mathbf{D}}_\alpha \mathbf{A} \overline{\mathbf{D}}_\alpha \mathbf{p}_\alpha + 2\mathbf{a}^\top \overline{\mathbf{D}}_\alpha \mathbf{p}_\alpha \mathbf{a} |\mathbf{D}_\alpha| + a_0 |\mathbf{D}_\alpha|^2 \;=\; 0,$$

where $\overline{\mathbf{D}}_\alpha$ denotes the adjoint and $|\mathbf{D}_\alpha|$ the determinant of $\mathbf{D}_\alpha$.

Degree Complexity

If we denote

- the i-th diagonal element of $\mathbf{D}_\alpha$ by d_i, i.e. $d_i := 1 + \alpha A_i$ and
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Example 3 (Point vs Central Surface) *For ellipsoids and hyperboloid we have:*

$$\mathbf{a} = 0 \quad \Rightarrow \quad \mathbf{p}_\alpha = \mathbf{p} \quad \textit{and}$$

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Summary: Point-Surface-Case

Point - Central Surface		
Ellipsoid	Hyperboloids	Cone
6	6	4

Point - Non-Central Surface		
Paraboloids	Elliptic / Hyperbolic Cylinder	Parabolic Cylinder
5	4	3

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Substituting $\mathbf{p}$ by the explicit representation of a conic, i.e.

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and in contrast to the point-surface case a bivariate system of equations:

$$\begin{aligned}
f(\alpha, t) &= \mathbf{p}_\alpha^T \overline{\mathbf{D}}_\alpha \mathbf{A} \overline{\mathbf{D}}_\alpha \mathbf{p}_\alpha + 2\mathbf{a}^T \overline{\mathbf{D}}_\alpha \mathbf{p}_\alpha \mathbf{a} |\mathbf{D}_\alpha| + a_0 |\mathbf{D}_\alpha|^2 = 0, \\
g(\alpha, t) &= \left(\overline{\mathbf{D}}_\alpha \mathbf{p}_\alpha - |\mathbf{D}_\alpha| \mathbf{p}\right) \frac{\partial \mathbf{p}}{\partial t} = 0.
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Example: Ellipse vs Ellipsoid

In the case of ellipsoids our system simplifies to

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f(\alpha, t) &= A_1 p_1^2 d_2^2 d_3^2 + A_2 p_2^2 d_1^2 d_3^2 + A_3 p_3^2 d_1^2 d_2^2 + a_0 d_1^2 d_2^2 d_3^2 = 0, \\
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For P being an ellipse we have to use a rational parameterization:

$$\mathbf{p}(t) := \mathbf{c} + \frac{1-t^2}{1+t^2}\mathbf{u} + \frac{2t}{1+t^2}\boldsymbol{v} \qquad \mathbf{p}'(t) := \frac{2}{1+t^2}\left[\frac{1-t^2}{1+t^2}\boldsymbol{v} - \frac{2t}{1+t^2}\mathbf{u}\right]$$

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Problem: The degree of the $\mathrm{Res}(f, g)$ is 32 (Mixed-Volume Bound).

Factorization of the Resultant Polynomial (I)

Proposition 1 (The Case of Central Surfaces)
Let $f = g = 0$ *be our system of equations, i.e.*

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(i) The pair (α_i, t_i) *is a solution of the bivariate system for every* t_i *solving the equation* $p_i = 0$*,* $i = 1, 2, 3$*.*

Factorization of the Resultant Polynomial (I)

Proposition 1 (The Case of Central Surfaces)
Let $f = g = 0$ *be our system of equations, i.e.*

$$\begin{aligned} f(\alpha, t) &= A_1 p_1^2 d_2^2 d_3^2 + A_2 p_2^2 d_1^2 d_3^2 + A_3 p_3^2 d_1^2 d_2^2 + a_0 d_1^2 d_2^2 d_3^2 &= 0, \\ g(\alpha, t) &= A_1 p_1 p_1' d_2 d_3 + A_2 p_2 p_2' d_1 d_3 + A_3 p_3 p_3' d_1 d_2 &= 0, \end{aligned}$$

and let α_i *denote the root of* d_i, $i = 1, 2, 3$. *Then the following holds:*

(i) The pair (α_i, t_i) *is a solution of the bivariate system for every* t_i *solving the equation* $p_i = 0$, $i = 1, 2, 3$.

(ii) If the curve is not a line, then every α_i *is a root of multiplicity 4 in* $\mathrm{Res}(f, g, t)$, *whereas every* t_i *has multiplicity 2 in* $\mathrm{Res}(f, g, \alpha)$.

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(ii) If the curve is not a line, then every α_i *is a root of multiplicity 4 in* $\mathrm{Res}(f, g, t)$*, whereas every* t_i *has multiplicity 2 in* $\mathrm{Res}(f, g, \alpha)$.

(iii) If the curve is a line, then every α_i *and* t_i *(which is now unique) is a root of multiplicity 2 in* $\mathrm{Res}(f, g, t)$ *and* $\mathrm{Res}(f, g, \alpha)$ *respectively.*

Factorization of the Resultant Polynomial (II)

Corollary 1 (Degree Complexity in the Case of Central Surfaces)
If the curve is not a line, the Resultant Polynomial can be written as the following product:

$$\mathrm{Res}(f, g, t) = h_\alpha \prod_{i=1}^{3} d_i^4 = h_\alpha \prod_{i=1}^{3} (\alpha - \alpha_i)^4,$$

$$\mathrm{Res}(f, g, \alpha) = h_t \prod_{i=1}^{3} p_i^2 = h_t \prod_{i=1}^{3} (t - t_{i1})^2 (t - t_{i2})^2.$$

where h_α *and* h_t *are univariate polynomials of degree at most 20.*

Factorization of the Resultant Polynomial (II)

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where h_α and h_t are univariate polynomials of degree at most 20.

Remark 2 (The Case of Non-Central Surfaces)
An analogous result can be found for non-central surfaces. However, in this case there are only two $d_i \neq 0$, $(i = 1, 2)$ and consequently only two α_i dividing $\mathrm{Res}(f, g, t)$.

Summary: Curve - Surface Case

Curve - Central Surface			
	Ellipsoid	Hyperboloids	Cone
Ellipse	20	20	12
Hyperbola	20	20	12
Parabola	14	14	8
Line	4	4	2

Curve - Non-Central Surface			
	Paraboloids	Elliptical / Hyperbolical Cylinder	Parabolical Cylinder
Ellipse	16	12	8
Hyperbola	16	12	8
Parabola	11	8	5
Line	3	2	1

The Surface-Surface Case

By setting up the LAGRANGE formalism for the problem

$$\min\,(\mathbf{x}-\mathbf{y})^2, \quad \mathbf{x} \in Q_1, \mathbf{y} \in Q_2$$

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we get the LAGRANGE function $\mathcal{L}$ and conditions $(i), \ldots, (iv)$:

$$\begin{aligned}\mathcal{L}(\mathbf{x}, \mathbf{y}; \alpha, \beta) &= (\mathbf{x}-\mathbf{y})^2 + \alpha(\mathbf{x}^T\mathbf{A}\mathbf{x} + 2\mathbf{a}^T\mathbf{x} + a_0) \\ &\quad + \beta(\mathbf{y}^T\mathbf{B}\mathbf{y} + 2\mathbf{b}^T\mathbf{y} + b_0)\end{aligned}$$

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(i)\ \partial\frac{\mathcal{L}(.)}{\partial \mathbf{x}} &= 0 \quad \Longleftrightarrow \quad \alpha(\mathbf{A}\mathbf{x} + \mathbf{a}) = \mathbf{y} - \mathbf{x} \\
(ii)\ \partial\frac{\mathcal{L}(.)}{\partial \mathbf{y}} &= 0 \quad \Longleftrightarrow \quad \beta(\mathbf{B}\mathbf{y} + \mathbf{b}) = \mathbf{x} - \mathbf{y}
\end{aligned}$$

The Surface-Surface Case

By setting up the LAGRANGE formalism for the problem

$$\min\,(\mathbf{x}-\mathbf{y})^2, \quad \mathbf{x}\in Q_1, \mathbf{y}\in Q_2$$

we get the LAGRANGE function $\mathcal{L}$ and conditions $(i), \ldots, (iv)$:

$$\begin{aligned}
\mathcal{L}(\mathbf{x},\mathbf{y};\alpha,\beta) &= (\mathbf{x}-\mathbf{y})^2 + \alpha(\mathbf{x}^T\mathbf{A}\mathbf{x} + 2\mathbf{a}^T\mathbf{x} + a_0) \\
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(ii)\ \partial\frac{\mathcal{L}(.)}{\partial\mathbf{y}} &= 0 \quad \Longleftrightarrow \quad \beta(\mathbf{B}\mathbf{y}+\mathbf{b}) = \mathbf{x}-\mathbf{y} \\
(iii)\ \partial\frac{\mathcal{L}(.)}{\partial\alpha} &= 0 \quad \Longleftrightarrow \quad \mathbf{x}^T A\mathbf{x} + 2\mathbf{a}^T\mathbf{x} + a_0 = 0 \\
(iv)\ \partial\frac{\mathcal{L}(.)}{\partial\beta} &= 0 \quad \Longleftrightarrow \quad \mathbf{y}^T B\mathbf{y} + 2\mathbf{b}^T\mathbf{y} + b_0 = 0
\end{aligned}$$

Solving the Lagrange System

By setting $\lambda := 1/\alpha$ and $\mu := 1/\beta$ we can derive from (i) and (ii):

$$\mathbf{x} = -(\mathbf{B}\mathbf{A} + \lambda\mathbf{B} + \mu\mathbf{A})^{-1}(\mathbf{B}\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{a}) =: -\frac{\overline{\mathbf{C}}_{\lambda,\mu}}{|\mathbf{C}_{\lambda,\mu}|}\mathbf{c}_B,$$

$$\mathbf{y} = -(\mathbf{A}\mathbf{B} + \lambda\mathbf{B} + \mu\mathbf{A})^{-1}(\mathbf{A}\mathbf{b} + \lambda\mathbf{b} + \mu\mathbf{a}) =: -\frac{\overline{\mathbf{C}}_{\lambda,\mu}^{T}}{|\mathbf{C}_{\lambda,\mu}|}\mathbf{c}_A,$$

where $\overline{\mathbf{C}}_{\lambda,\mu}$ and $|\mathbf{C}_{\lambda,\mu}|$ are (matrix) polynomials in λ and μ.

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$$\mathbf{y} = -(\mathbf{AB} + \lambda\mathbf{B} + \mu\mathbf{A})^{-1}(\mathbf{A}\mathbf{b} + \lambda\mathbf{b} + \mu\mathbf{a}) =: -\frac{\overline{\mathbf{C}}^{T}_{\lambda,\mu}}{|\mathbf{C}_{\lambda,\mu}|}\mathbf{c}_A,$$

where $\overline{\mathbf{C}}_{\lambda,\mu}$ and $|\mathbf{C}_{\lambda,\mu}|$ are (matrix) polynomials in λ and μ.

Substituting $\mathbf{x}$ and $\mathbf{y}$ in (iii) and (iv) and multiplying my the denominator, gives the system:

$$f(\lambda,\mu) = \mathbf{c}_B^T\overline{\mathbf{C}}^T_{\lambda,\mu}\mathbf{A}\overline{\mathbf{C}}_{\lambda,\mu}\mathbf{c}_B - 2|\mathbf{C}_{\lambda,\mu}|\mathbf{a}^T\overline{\mathbf{C}}_{\lambda,\mu}\mathbf{c}_B + a_0|\mathbf{C}_{\lambda,\mu}|^2 = 0,$$

$$g(\lambda,\mu) = \mathbf{c}_A^T\overline{\mathbf{C}}_{\lambda,\mu}\mathbf{B}\overline{\mathbf{C}}^T_{\lambda,\mu}\mathbf{c}_A - 2|\mathbf{C}_{\lambda,\mu}|\mathbf{b}^T\overline{\mathbf{C}}^T_{\lambda,\mu}\mathbf{c}_A + b_0|\mathbf{C}_{\lambda,\mu}|^2 = 0,$$

The Inverse of $\mathbf{C}_{\lambda,\mu}$

Lemma 1
The adjoint and determinant of $\mathbf{C}_{\lambda,\mu} = \mathbf{BA} + \lambda\mathbf{B} + \mu\mathbf{A}$ *is given by*

$$\begin{aligned} \overline{\mathbf{C}_{\lambda,\mu}} &= \overline{\mathbf{B}}\lambda^2 + \overline{\mathbf{A}}\mu^2 + \mathsf{T}_A\overline{\mathbf{B}}\lambda + \overline{\mathbf{A}}\mathsf{T}_B\mu + (\mathsf{T}_B\mathsf{T}_A - \mathsf{T}_{AB})\lambda\mu + \overline{\mathbf{A}}\,\overline{\mathbf{B}}, \\ |\mathbf{C}_{\lambda,\mu}| &= |\mathbf{B}|\lambda^3 + |\mathbf{A}|\mu^3 + |\mathbf{B}|tr(\mathbf{A})\lambda^2 + |\mathbf{A}|tr(\mathbf{B})\mu^2 + \\ &\quad |\mathbf{B}|tr(\overline{\mathbf{A}})\lambda + |\mathbf{A}|tr(\overline{\mathbf{B}})\mu + tr(\overline{\mathbf{B}}\mathbf{A})\lambda^2\mu + tr(\overline{\mathbf{A}}\mathbf{B})\lambda\mu^2 + \\ &\quad \left(tr(\overline{\mathbf{A}})tr(\overline{\mathbf{B}}) - tr(\overline{\mathbf{A}}\,\overline{\mathbf{B}})\right)\lambda\mu + |\mathbf{A}||\mathbf{B}|, \end{aligned}$$

where $\mathsf{T}_M := tr(\mathbf{M})\mathsf{E} - \mathbf{M}$ *for a matrix* $\mathbf{M} \in \mathbb{R}^{3\times 3}$.

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&\quad \big(tr(\overline{\mathbf{A}})tr(\overline{\mathbf{B}}) - tr(\overline{\mathbf{A}}\,\overline{\mathbf{B}})\big)\lambda\mu + |\mathbf{A}||\mathbf{B}|,
\end{aligned}$$

where $\mathsf{T}_M := tr(\mathbf{M})\mathsf{E} - \mathbf{M}$ *for a matrix* $\mathbf{M} \in \mathbb{R}^{3\times 3}$.

Proposition 2 (Bivariate Degree Complexity)
The polynomials f *and* g *have degree 6 in* λ *as well as* μ.
Moreover the total degree of f *and* g *is also 6.*

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Corollary 2 (Bezout)
The degree of the resultant polynomial $\mathrm{Res}(f, g)$ *is at most 36.*

Factorization of the Resultant Polynomial

Lemma 2
Let $f = g = 0$ *be our system of polynomial equations, i.e.*

$$
\begin{aligned}
f(\lambda,\mu) &= \mathbf{c}_B^{\mathsf{T}}\overline{\mathbf{C}}_{\lambda,\mu}^{\mathsf{T}}\mathbf{A}\overline{\mathbf{C}}_{\lambda,\mu}\mathbf{c}_B - 2|\mathbf{C}_{\lambda,\mu}|\mathbf{a}^{\mathsf{T}}\overline{\mathbf{C}}_{\lambda,\mu}\mathbf{c}_B + a_0|\mathbf{C}_{\lambda,\mu}|^2 = 0,\\
g(\lambda,\mu) &= \mathbf{c}_A^{\mathsf{T}}\overline{\mathbf{C}}_{\lambda,\mu}\mathbf{B}\overline{\mathbf{C}}_{\lambda,\mu}^{\mathsf{T}}\mathbf{c}_A - 2|\mathbf{C}_{\lambda,\mu}|\mathbf{b}^{\mathsf{T}}\overline{\mathbf{C}}_{\lambda,\mu}^{\mathsf{T}}\mathbf{c}_A + b_0|\mathbf{C}_{\lambda,\mu}|^2 = 0,
\end{aligned}
$$

and the system $\mathbf{h}$ *be defined as follows:*

$$\mathbf{h}(\lambda,\mu) := (h_1, h_2, h_3)^{\mathsf{T}} = \overline{\mathbf{C}}_{\lambda,\mu}\mathbf{c}_B = \mathbf{0}.$$

Then the common roots of the polynomials $r_{ij} := \mathrm{Res}(h_i, h_j)$*,* $1 \leq i < j \leq 3$*, solve* $\mathrm{Res}(f, g)$ *with multiplicity 4.*

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Then the common roots of the polynomials $r_{ij} := \mathrm{Res}(h_i, h_j)$, $1 \leq i < j \leq 3$, *solve* $\mathrm{Res}(f, g)$ *with multiplicity 4.*

Proposition 3 (Degree Complexity)
Let p *denote the polynomial given by the common roots of* r_{ij}, $1 \leq i < j \leq 3$, *and their multiplicities in* $\mathrm{Res}(f, g)$. *Then the remaining polynomial* $\mathrm{Res}(f, g)/p$ *is of a degree of at most 24.*

Summary: Surface-Surface Case

	Central Surfaces		Non-Central Surfaces		
	Ellipsoid / Hyperboloids	Cone	Paraboloids	E. / H. Cylinder	Parabolic Cylinder
Ellipsoids / Hyperboloid	24	12	18	12	8
Cone		4	8	4	2
Paraboloids			13	8	5
E. / H. Cylinder				4	2

The Point-Curve Case

Let the conic Q lie in the $x_1 x_2$-plane and be centered around the origin, i.e.

$$Q : \quad \mathbf{q}(t) = r(t)\mathbf{u} + s(t)\boldsymbol{v}, \quad \mathbf{u}^T\boldsymbol{v} = 0.$$

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Projecting the query point $\mathbf{p}$ onto the same plane yields a 2-D problem:

$$\min_t (\overline{\mathbf{p}} - r(t)\mathbf{u} - s(t)\boldsymbol{v})^2.$$

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$$\min_t (\overline{\mathbf{p}} - r(t)\mathbf{u} - s(t)\boldsymbol{v})^2.$$

Setting the derivative of the distance function equal to zero, gives

$$f(t) \;=\; r(t)r'(t)\mathbf{u}^2 + s(t)s'(t)\boldsymbol{v}^2 - r'(t)\overline{\mathbf{p}}^T\mathbf{u} - s'(t)\overline{\mathbf{p}}^T\boldsymbol{v} = 0,$$

with $r' \equiv \frac{dr}{dt}$ and $s' \equiv \frac{ds}{dt}$.

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Let the conic Q lie in the x_1x_2-plane and be centered around the origin, i.e.

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with $r' \equiv \frac{dr}{dt}$ and $s' \equiv \frac{ds}{dt}$.

Example 4 (Point vs Ellipse) *For ellipses we have:*

$$f(t) \;=\; \overline{\mathbf{p}}^T\boldsymbol{v}t^4 + 2\left[\overline{\mathbf{p}}^T\mathbf{u} - (\boldsymbol{v}^2 - \mathbf{u}^2)\right]t^3 + 2\left[\overline{\mathbf{p}}^T\mathbf{u} + (\boldsymbol{v}^2 - \mathbf{u}^2)\right]t^2 - \overline{\mathbf{p}}^T\boldsymbol{v} \;=\; 0.$$

The Curve-Curve Case

Given two conics P and Q, i.e.

$$
\begin{aligned}
P: \quad \mathbf{p}(t_1) &= \quad r_1(t_1)\mathbf{u}_1 + s_1(t_1)\boldsymbol{v}_1, \quad \mathbf{u}_1^T\boldsymbol{v}_1 = 0,\\
Q: \quad \mathbf{q}(t_2) &= \mathbf{c}_2 + r_2(t_2)\mathbf{u}_2 + s_2(t_2)\boldsymbol{v}_2, \quad \mathbf{u}_2^T\boldsymbol{v}_2 = 0.
\end{aligned}
$$

The partial derivatives of $\delta^2(t_1, t_2) = (\mathbf{q}(t_2) - \mathbf{p}(t_1))^2$ yield the following system of bivariate equations:

$$
\begin{aligned}
f(t_1, t_2) &= [\mathbf{q}(t_2) - \mathbf{p}(t_1)]^T \left[-\frac{\partial r_1}{\partial t_1}\mathbf{u}_1 - \frac{\partial s_1}{\partial t_1}\boldsymbol{v}_1 \right] = 0,\\
g(t_1, t_2) &= [\mathbf{q}(t_2) - \mathbf{p}(t_1)]^T \left[\frac{\partial r_2}{\partial t_2}\mathbf{u}_2 + \frac{\partial s_2}{\partial t_2}\boldsymbol{v}_2 \right] = 0.
\end{aligned}
$$

Example: Distance Between Two Ellipses

If P and Q are both ellipses, we can write our conditions as:

$$\begin{aligned} f(t_1, t_2) &= (1 + t_1^2) f_1(t_1, t_2) + (1 + t_2^2) f_2(t_1), \\ g(t_1, t_2) &= (1 + t_1^2) g_1(t_2) + (1 + t_2^2) g_2(t_1, t_2), \end{aligned}$$

where f_i and g_i, $i = 1, 2$, are polynomials of degrees at most 2 in t_1 and t_2.

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where f_i and g_i, $i = 1, 2$, are polynomials of degrees at most 2 in t_1 and t_2.

Observation 1
Every $(\xi_1, \xi_2) \in \{-i, i\}^2$ *solves the bivariate system and hence,* $(1 + t_1^2)^2$ *is a factor of* $\mathrm{Res}(f, g, t_2)$.

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Every $(\xi_1, \xi_2) \in \{-i, i\}^2$ *solves the bivariate system and hence,* $(1 + t_1^2)^2$ *is a factor of* $\mathrm{Res}(f, g, t_2)$.

Proposition 4 (Degree Complexity)
The distance between two ellipses can be computed by solving polynomials of a degree of at most 16.

Proof.
The proposition follows from the fact that the mixed-volume of f and g bounds the degree of $\mathrm{Res}(f, g)$ by 20.

Summary: Point-Curve and Curve-Curve Cases

Point - Curve and Curve-Curve				
	Ellipse	Hyperbola	Parabola	Line
Point	4	4	3	1
Ellipse	16	16	12	4
Hyperbola		16	12	4
Parabola			9	3
Line				1

Natural Conics, Quadrics and the Torus

	Natural Conics			Natural Quadrics				Torus
	Point	Line	Circle	Plane	Sphere	C. Cylinder	C. Cone	Torus
Point	1	1	2	1	2	2	2	2
Line		1	4	1	2	2	2	4
Circle			8	2	2	4	8	8
Plane				1	1	1	2	2
Sphere					2	2	2	2
Circular Cylinder						2	2	4
Circular Cone							4	8
Torus								8

Natural Conics, Quadrics and the Torus

	Natural Conics			Natural Quadrics				Torus
	Point	Line	Circle	Plane	Sphere	C. Cylinder	C. Cone	Torus
Point	1	1	2	1	2	2	2	2
Line		1	4	1	2	2	2	4
Circle			8	2	2	4	8	8
Plane				1	1	1	2	2
Sphere					2	2	2	2
Circular Cylinder						2	2	4
Circular Cone							4	8
Torus								8

Remark 3 (Optimality)
It is shown in [Farouki,Neff,O'Connor89] that a degree of 8 is a lower bound on the degree complexity in the circle-circle case. Hence, it is proved that the upper bound result is strict for this special class of surfaces.